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Study of a Bianchi type-V cosmological model with torsion

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A Bianchi type-V exact solution of the Einstein-Cartan theory representing several classes of perfect-fluid homogeneous cosmological models is studied. With the introduction of an effective energy-momentum tensor, a dual description of the solution is given within the framework of general relativity, whereby it is shown that the two theories, although formally equivalent, provide physically inequivalent interpretations of the solution. Examined in some detail are the singularity behavior as well as other features of the models.

I. INTRODUCTION

As it now stands, the so-called Einstein-Cartan (-Sciama-Kibble) (EC) theory is among the best candidates for a classical gauge theory of the Poincaré group.[1] It is a natural extension of Einstein's general relativity (GR) theory. Allowing the spacetime to carry torsion, it successfully incorporates spin and supplies a mechanism which may prevent the occurrence of singularities in cosmological models. The two theories are formally equivalent due to the fact that the algebraic relation between spin and torsion in EC theory allows one to recast its field equations in the form of Einstein's field equations. Specific aspects of EC theory as well as the exact nature of its relation to GR can be conveniently studied on the basis of exact solutions. In this paper we study an exact solution to the EC field equations representing a spatially homogeneous spacetime, filled with a perfect spinning fluid. When the field equations are recast in the form of Einstein's field equations, the resulting effective energy-momentum tensor is still that of a fluid, which, however, due to the appearance of viscosity is no longer perfect and whose energy density may even become negative as in the case of nonsingular solutions.

In the following section we set up the field equations and give the exact EC solution representing several cosmological models. In Sec. III we present the equivalent GR picture which involves the introduction of an effective energy-momentum tensor, representing a nonperfect (viscous) fluid. Based on this result we point out the physical inequivalence between EC theory and GR in the last section, where we also discuss several other features of the models.

II. THE EXACT EC SOLUTION

For details on the formalism, notation, etc., see Ref. 2. All quantities refer to the orthonormal tetrad $\{\omega^\alpha\}$ with $\omega^0=dt$, $\omega^1=a\sigma^1$, $\omega^2=\lambda a(\sigma^2+\nu\sigma^3)$, $\omega^3=\mu a\sigma^3$, and $g_{\alpha\beta}=\mathrm{diag}(-1,1,1,1)$, where a,λ,μ,ν are functions of the cosmic time t. The left-invariant one-forms σ^i span the hypersurfaces of homogeneity which are generated by a Bianchi type-V group of motions with $d\sigma^1=0$, $d\sigma^2=\sigma^1\wedge\sigma^2$, $d\sigma^3=\sigma^1\wedge\sigma^3$. The EC field equations are

$$R^\alpha{}_\beta-\tfrac{1}{2}\delta^\alpha{}_\beta R=t^\alpha{}_\beta\ , \tag{1}$$

$$T^\alpha{}_{\beta\gamma}-\delta^\alpha{}_\beta T^\epsilon{}_{\epsilon\gamma}-\delta^\alpha{}_\gamma T^\epsilon{}_{\beta\epsilon}=S^\alpha{}_{\beta\gamma}\ . \tag{2}$$

We choose the classical description of spin tensor, namely, $S^\alpha{}_{\beta\gamma}=S_{\beta\gamma}u^\alpha$ where u^α is the four-velocity of the perfect fluid content, orthogonal to the hypersurfaces of homogeneity. The energy-momentum tensor is then given by $t_{\alpha\beta}=\mathrm{diag}(\rho_C,p_C,p_C,p_C)$, where ρ_C is the energy density and $p_C=(1-\gamma)\rho_C$ the isentropic pressure of the cosmic fluid. We further choose $S_{23}=-S_{32}\equiv 2\omega$ to be the only nonvanishing components of the spin density, corresponding to a spin aligned along the ω^1 direction. The algebraic part of the field equation, Eq. (2), then reads $T^\alpha{}_{\beta\gamma}=S^\alpha{}_{\beta\gamma}=S_{\beta\gamma}u^\alpha$ and supplies $T^0{}_{23}=-T^0{}_{32}=2\omega$ as the only nonvanishing components of the torsion. The rest of the field equations are

$$\begin{aligned}&R_{00}=\tfrac{1}{2}(\rho_C+3p_C)\ ,\\&R_{11}=R_{22}=R_{33}=\tfrac{1}{2}(\rho_C-p_C)\ ,\\&R_{0j}=R_{ij}=0\ .\end{aligned} \tag{3}$$

For completeness we give below the result of our calculation for the components of the Ricci tensor which are not identically zero (in the sequel, ω^2 should not be confused with the one-form ω^2):

$$-R_{00}=(\ln\lambda\mu a^3)^{\cdot\cdot}+(\ln a)^{\cdot 2}+(\ln\lambda a)^{\cdot 2}+(\ln\mu a)^{\cdot 2}-2\omega^2+2\varphi^2 , \quad (4a)$$

$$R_{11}=(\ln a)^{\cdot\cdot}+(\ln\lambda\mu a^3)^{\cdot}(\ln a)^{\cdot}-2a^{-2} , \quad (4b)$$

$$R_{22}=(\ln\lambda a)^{\cdot\cdot}+(\ln\lambda\mu a^3)^{\cdot}(\ln\lambda a)^{\cdot}-2a^{-2}-2\varphi(\omega+\varphi) , \quad (4c)$$

$$R_{33}=(\ln\mu a)^{\cdot\cdot}+(\ln\lambda\mu a^3)^{\cdot}(\ln\mu a)^{\cdot}-2a^{-2}+2\varphi(\omega+\varphi) , \quad (4d)$$

$$R_{01}=R_{10}=-a^{-1}(\ln\lambda\mu)^{\cdot} , \quad (4e)$$

$$R_{23}=\varphi(\ln\varphi\lambda^2a^3)^{\cdot}-\omega(\ln\omega\mu^2a^3)^{\cdot} , \quad (4f)$$

$$R_{32}=\varphi(\ln\varphi\lambda^2a^3)^{\cdot}+\omega(\ln\omega\lambda^2a^3)^{\cdot} , \quad (4g)$$

where a dot denotes d/dt and $\varphi\equiv\frac{1}{2}\lambda\mu^{-1}\dot{\nu}$. From $R_{01}=0$ it follows that $\lambda\mu$ must be a constant; without loss of generality we choose

$$\lambda\mu=1 . \quad (5)$$

From $R_{23}=R_{32}=0$ we obtain (Λ,Ω are constants)

$$\omega=\Omega a^{-3} , \quad (6)$$

$$\omega+\varphi=\Lambda\lambda^{-2}a^{-3} . \quad (7)$$

Equation (6) expresses conservation of the spin density. A similar conservation equation for the energy density follows from $t_\alpha{}^\beta{}_{;\beta}=0$, which holds in our case, namely (with M a constant)

$$\rho_C=3Ma^{-3(2-\gamma)} . \quad (8)$$

With τ a new time coordinate defined by $d\tau\equiv a^{-3}dt$ and A a constant, we obtain from the 11 field equation

$$\int (a^3+Ma^{3\gamma}-\tfrac{1}{3}A)^{-1/2}a^{-1}da=\tau . \quad (9)$$

From $R_{22}=R_{33}$ and using (6) and (7), we obtain with one integration (and with a prime denoting $d/d\tau$)

$$(\ln\lambda)'=-\Lambda^2\lambda^{-4}+2\Lambda\Omega\lambda^{-2}-A , \quad (10)$$

where the constant of integration turns out to equal $-A$. This follows from a substitution of (6)–(10) into the super-Hamiltonian constraint equation $G_{00}=t_{00}$, which is then satisfied identically.

The solution obtained so far, as expressed by Eqs. (5)–(10), satisfies all the field equations and it is the most general as it has been obtained by quadratures. To proceed with the explicit form of λ,ν as functions of τ (a cannot in general be given in closed form) it will be illuminating to first give the shear scalar $\sigma^2=\frac{1}{2}\sigma^{\alpha\beta}\sigma_{\alpha\beta}$. A straightforward calculation gives the following nonzero components of the shear tensor: $\sigma_{22}=-\sigma_{33}=(\ln\lambda)'a^{-3}$, $\sigma_{23}=\sigma_{33}=\varphi$ [for completeness we mention here that the volume expansion is $(\ln a^3)^{\cdot}$]. It now follows that $\sigma^2=(\Omega^2-A)a^{-6}$, which, in analogy with (6), we write as

$$\sigma=\Sigma a^{-3} . \quad (11)$$

Thus, the constant

$$A=\Omega^2-\Sigma^2 \quad (12)$$

is a measure of the dominance of spin density over shear or vice versa. As discussed in more detail in the last section, we see from (9) that $(\ln a)'=(a^4+Ma^{3\gamma}-\frac{1}{3}A)^{1/2}$ becomes zero for some $a=a_b>0$ when $A>0$ (dominance of spin density) giving generally a "bouncing" solution. In all other cases a singularity occurs at $a=0$. Depending on the sign of A we obtain now the following expressions from (10):

$$A>0: \lambda^2[A^{-1}\Lambda\Omega](1-\Sigma\Omega^{-1}\cos 2\sqrt{A}\,\tau) , \quad (13a)$$

$$A=0: \lambda^2=[\tfrac{1}{2}\Lambda\Omega^{-1}][1+(2\Omega\tau)^2) , \quad (13b)$$

$$A<0: \lambda^2=[\,|A|^{-1}\Lambda\Omega](-1+\Sigma\Omega^{-1}\cosh 2\sqrt{|A|}\,\tau) . \quad (13c)$$

Finally, we have from (7) $\nu'=2(\Lambda-\Omega\lambda^2)\lambda^{-4}$ where we substitute $d/d\tau$ in terms of $d/d\lambda$ using (10) and integrate to get

$$\nu^2=-\lambda^4+2\Lambda^{-1}\Omega\lambda^{-2}-A\Lambda^{-2} . \quad (14)$$

The above solution is not new: it turns out to be effectively the same as the one given by Taffel.[3] One can show that our result coincides with Taffel's (notice that his $\theta^1,\theta^2,\theta^3$ correspond to our $\sigma^2,\sigma^3,\sigma^1$, respectively) if a proper choice for Λ is made (see below) and if the origin of τ is shifted to change the cosine into sine (for the case $A>0$) or the σ^2,σ^3 transformed to $\sigma^2\to\sigma^2-\sigma^3,\sigma^3\to\sigma^2+\sigma^3$ (for $A<0$). The last transformation is a particular case of constant automorphisms of the form $\sigma^i\to\tau^i=A^i{}_j\sigma^j$ for (constant) A_{ij} such that the new invariant one-forms τ^i still characterize a Bianchi type-V group of motions. In fact Λ can be thought of as arising by such an automorphism and, being redundant in that sense, can be assigned any value, as, for example, to make the quantities in square brackets in (13) equal to 1.

III. EQUIVALENT GR DESCRIPTION

In order to study the equivalent GR picture of the model we introduce an effective energy-momentum tensor $T_{\alpha\beta}$, defined by

$$\hat{R}_{\alpha\beta}=T_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}T\ , \tag{15}$$

where $\hat{R}_{\alpha\beta}$ is the Riemannian Ricci tensor of our spacetime, corresponding to the Levi-Civita part of the connection. Its components are, in effect, given by Eqs. (4) if we set $\omega=0$. It then follows from (15) that

$$\begin{aligned}
&T_{00}=\rho_C-\omega^2\ ,\\
&T_{11}=p_C-\omega^2\ ,\\
&T_{22}=p_C-(3-2\Omega^{-1}\Lambda\lambda^{-2})\omega^2\ ,\\
&T_{33}=p_C+(1-2\Omega^{-1}\Lambda\lambda^{-2})\omega^2\ ,\\
&T_{23}=T_{32}=-2\Omega^{-1}\Lambda(\nu+N)\omega^2\ .
\end{aligned} \tag{16}$$

This result shows that the effective energy-momentum tensor $T_{\alpha\beta}$ is that of a viscous fluid. To see this, we compare (16) with the general expression[4]

$$T_{\alpha\beta}=\rho u_\alpha u_\beta+ph_{\alpha\beta}+\pi_{\alpha\beta}+u_\alpha q_\beta+u_\beta q_\alpha \tag{17}$$

for the energy-momentum tensor of a nonperfect fluid with energy density ρ, scalar pressure p, anisotropic pressure $\pi_{\alpha\beta}$, and energy-flux density q_α. These quantities assume in our case the following values:

$$\begin{aligned}
&\rho=\rho_C-\omega^2\ ,\\
&p=p_C-\omega^2\ ,\\
&\pi_{22}=-\pi_{33}=2(\Omega^{-1}\Lambda\lambda^{-2}-1)\omega^2\ ,\\
&\pi_{23}=\pi_{32}=-2\Omega^{-1}\Lambda(\nu+N)\omega^2\ ,
\end{aligned} \tag{18}$$

where the rest of the $\pi_{\alpha\beta}$ components as well as the energy-flux density vanish.

It is clear from (18) that the presence of spin in the original EC model is reflected in the GR picture in the appearance of anisotropic pressure proportional to ω^2 and the reduction of the energy density and pressure values by ω^2. In other words, we observe the appearance of kinetic viscosity and bulk viscosity, respectively. Notice however that the usual phenomenological relation $\pi_{\alpha\beta}=-\lambda\sigma_{\alpha\beta}$ holds here with a tensorial viscosity coefficient λ rather than the usual scalar, due to the relation $\pi_{\alpha\beta}\pi^{\alpha\beta}=0$ which holds in our case.

IV. DISCUSSION

Let us now look again at the nonsingular case in the context of the GR picture. Using Eqs. (6), (8), (10), and (11) and assuming $\gamma\neq 0$ we can write $\rho_C=(a/a_0)^{3\gamma}(\omega^2-\sigma^2)$ where $a_0{}^{3\gamma}\equiv\frac{1}{3}AM^{-1}$. Then (18) gives

$$\rho=[(a/a_0)^{3\gamma}-1]\omega^2-(a/a_0)^{3\gamma}\sigma^2\ . \tag{19}$$

The meaning of the constant $a_0{}^{3\gamma}$ is easily determined from (14): it is slightly larger than $a_b{}^{3\gamma}$, where a_b is the minimum value of a (the "bounce radius"). Since a_b cannot in general be given in closed form, we may instead use a_0 for all practical purposes. As a approaches a_b the second term on the right-hand side of (19) will eventually dominate. At $a=(1-\Sigma^2\Omega^{-2})^{-1/(3\gamma)}a_0$, ρ will become zero, going into negative values, with $\rho=-\sigma^2$ at $a=a_0$. This result is of no surprise, of course, in view of the Penrose-Hawking singularity theorems,[5] but it underlines, contrary to earlier claims,[6] the physical *inequivalence* between EC theory and GR. A case of particular interest included in the above analysis is that of dust ($\gamma=1$). At the other extreme, $\gamma=0$, the case of stiff matter, the condition $A>0$ alone does not guarantee the avoidance of a singularity. Equation (19), valid for $\gamma\neq 0$, should now be replaced by

$$\rho=(3MA^{-1}-1)\omega^2-3MA^{-1}\sigma^2=(3M-\Omega^2)a^{-6}\ , \tag{20}$$

which shows that ρ has constant sign. We obtain a nonsingular model if $A>3M$. The minimum value of a is then $a_b=(\frac{1}{3}A-M)^{1/4}$, ρ is always negative and (14) integrates to

$$a=a_b(\cos 2a_b{}^2\tau)^{-1/2}\ . \tag{21}$$

For $A\leq 3M$ a singularity will always occur even though ρ may still be negative. This last possibility is realized if we further have $\Omega^2>3M$. It illustrates the fact that negative-energy density alone in the GR picture is not sufficient to guarantee the avoidence of a singularity. The spin must "overcome" the accumulative effect of both energy density *and* shear.

In view of the difficulty to obtain closed-form expressions in the general case we give below the asymptotic behavior of the model very close to and very far from the bounce. Close to the bounce (at $t=0$) we have $a=a_b+\frac{1}{4}[(4-3\gamma)a_b{}^{-1}+\gamma Aa_b{}^{-5}]t^2$ and $\tau=a_b{}^{-3}t$. Far away (as $t\to\pm\infty$), $a=|t|$ and $\tau=\tau_+-\frac{1}{2}t^{-2}$ (for $t\to+\infty$), $\tau=\tau_-+\frac{1}{2}t^{-2}$ (for

$t \to -\infty$) with $\tau_\pm$ constants. When the spin density vanishes, only the $A \leq 0$ part of the solution survives giving effectively the well-known Heckmann and Schucking (HS) model.[7] We note here the analogy in the mechanism of incorporating spin density to HS as compared to the addition of kinematic vorticity to HS, as done by Batakis and Cohen.[8] The analogy, interesting in itself, leads also to the following observation. Let us recall that the kinematic vorticity is usually described by the vorticity one-form Ω (not to be confused with the constant used in the solution)

$$\Omega = \frac{1}{2} * (u \wedge du) \tag{22}$$

which for a torsionless theory is the dual of the vector

$$\Omega^\alpha = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} u_\beta \omega_{\gamma\delta} , \tag{23}$$

where $\omega_{\gamma\delta}$ is the vorticity tensor. However, in our case the "kinematic" Ω of (22) is zero, while the Ω^α of (23) is not. Indeed the vorticity tensor turns out to be $\omega_{\alpha\beta} = \frac{1}{2} S_{\alpha\beta}$ and, consequently, (23) has one nonvanishing component, namely, $\Omega^1 = \omega$. This is actually of no surprise if we recall that the spin density was initially chosen so as to correspond to spin vectors aligned along the "1" direction.

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[1] F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester, Rev. Mod. Phys. 48, 393 (1976) and references cited therein.

[2] D. Tsoubelis, Phys. Rev. D 20, 3004 (1979).

[3] J. Taffel, Phys. Lett. 45A, 341 (1973).

[4] M. A. H. MacCallum, in *Cargèse Lectures in Physics*, edited by E. Schatzman (Gordon and Breach, New York, 1973), Vol. 6, p. 61.

[5] S. Hawking and R. Penrose, Proc. R. Soc. London A314, 529 (1970).

[6] J. M. Nester, Phys. Rev. D 16, 2395 (1977).

[7] O. Heckmann and E. Schucking, in *Gravitation*, edited by L. Witten (Wiley, New York, 1962).

[8] N. A. Batakis and J. M. Cohen, Phys. Rev. D 12, 1544 (1975).